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Particle Creation in a Cosmological Anisotropic Universe

Víctor M. Villalba[1]

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We analyze the phenomenon of particle creation in a cosmological anisotropic universe. We compute, via the Bogoliubov transformations, the density number of scalar and spin-1/2 particles created. We obtain that they are respectively described by Bose–Einstein and Fermi–Dirac distributions.

1. INTRODUCTION

Quantum processes in curved space-time are undoubtedly among the most interesting and puzzling problems in theoretical physics. After the appearance of the pioneering article by Hawking about pair production in the vicinity of black holes, a great body of papers have been published, mainly trying to understand the mechanism that gives rise to the thermal particle distribution and its relation to thermodynamics. It is noteworthy that Hawking's result was preceded by a series of articles where the question was to discuss particle production in cosmological universes (Parker, 1968; Zeldovich and Stardoinskii, 1971). Almost all of the work in this area deals with isotropic and homogeneous gravitational backgrounds, mainly in de Sitter and Robertson–Walker models, and only a few try to discuss quantum processes in anisotropic universes.

The study of quantum effects in gravitational backgrounds with initial singularities present an additional difficulty. The techniques commonly applied in order to define positive and negative particle frequencies fail, and a different approach is needed to circumvent the problem related to the initial singularity. In this direction, the Feynman path-integral method has been

[1]Centro de Física, Instituto Venezolano de Investigaciones Científicas IVIC, Caracas 1020-A Venezuela; e-mail: villalba@dino.conicit.ve, villalba@taquion.ivic.ve.

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applied to the quantization of a scalar field moving in the the Chitre–Hartle universe (Chitre and Hartle, 1977; Fischetti *et al.*, 1979), which is a Robertson–Walker background metric with scale factor $R(t) = t$. This model has a curvature singularity at $t = 0$ with the scalar curvature given by $R = 6/t^2$, and it is perhaps the best-known example where a time singularity appears and consequently any adiabatic prescription in order to define particle states fails. A spin-1/2 extension has been considered by Sahni (1984).

A different approach to the problem of classifying single-particle states on curved spaces is based on the idea of diagonalizing the Hamiltonian. This technique permits one to compute the mean number of particles produced by a singular cosmological model, and in particular by the Chitre–Hartle universe (Chitre and Hartle, 1977).

An interesting scenario for discussing particle creation processes is the anisotropic universe associated with the metric

$$ds^2 = -dt^2 + t^2(dx^2 + dy^2) + dz^2 \tag{1}$$

The line element (1) presents a timelike singularity at $t = 0$. The scalar curvature is $R = 2/t^2$, and consequently the adiabatic approach is not suitable for defining particle states. With the help of the Hamiltonian diagonalization method (Grib *et al.*, 1988), it has been possible (Bukhbinder, 1980) to compute the rate of scalar particles produced in the space with the metric (1), obtaining as a result a Bose–Einstein distribution. It is the purpose of the present article to show that a quasiclassical approach can give a prescription for the identification of positive- and negative-frequency states in the vicinity of time singularities. We are going to compute the mean number of scalar and spin-1/2 particles produced in the universe described by the metric (1). The idea behind the method is the following: First, we solve the relativistic Hamilton–Jacobi equation and, looking at its solutions, we identify positive- and negative-frequency modes. Second, we solve the Klein–Gordon equation and, after comparing with the results obtained for the quasiclassical limit, we identify the positive- and negative-frequency states. This technique has been shown to be of help in analyzing quantum effects in accelerated frames of reference (Costa, 1989; Percoco and Villalba, 1992).

The relativistic Hamilton–Jacobi equation can be written as

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} - m^2 = 0 \tag{2}$$

Since the metric $g_{\alpha\beta}$ given by (1) only depends on the time parameter t, the function S can be separated as

$$S = F(t) + k_x x + k_y y + k_z z \tag{3}$$

Substituting (3) into (2), we obtain

$$\dot{F}^2 = \frac{k_x^2\, 1 + k_y^2}{t^2} + k_z^2 + m^2 \tag{4}$$

The solution of equation (4) presents the asymptotic behavior

$$\lim_{t\to\infty} F = \pm\sqrt{k_z^2 + m^2}t, \qquad \Phi \to C\exp(\pm i\sqrt{k_z^2 + m^2}t) \tag{5}$$

as $t \to \infty$, and

$$\lim_{t\to 0} F = \pm\sqrt{k_x^2 + k_y^2}\log t, \qquad \Phi \to Ct^{\pm i(k_x^2+k_y^2)^{1/2}} \tag{6}$$

as $t \to 0$, that is, in the initial singularity. Notice that the time dependence of the relativistic wave function is obtained via the exponential operation $\Phi \to \exp(iS)$. Here it is worth mentioning that the behavior of positive- and negative-frequency states is selected depending on the sign of the operator $i\partial_t$. Positive-frequency modes will have positive eigenvalues and for negative-frequency states we will have the opposite. Then in equations (5) and (6) the upper signs are associated with negative-frequency values and the lower signs correspond to positive-frequency states. After making this identification we can analyze the solutions of the Klein–Gordon and Dirac equations in the background field (1).

The covariant generalization of the Klein–Gordon equation takes the form

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi - (m^2 + \xi R)\Phi = 0 \tag{7}$$

where ∇_α is the covariant derivative, R is the scalar curvature, and ξ is a dimensionless coupling constant which takes the value $\xi = 1/6$ in the conformal case, and $\xi = 0$ when a minimal coupling is considered. After substituting (1) into (7) we obtain

$$\frac{\partial^2\Phi_0}{\partial t^2} - \frac{\partial^2\Phi}{\partial z^2} - \frac{1}{t^2}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \left(m^2 + \frac{\xi}{t^2}\right)\Phi = 0 \tag{8}$$

Since equation (8) commutes with the operators $i\partial_x$, $i\partial_y$, $i\partial_z$, we have that the substitution

$$\Phi = t^{-1}\Phi_0 e^{i(k_x x + k_y y + k_z z)} \tag{9}$$

reduces equation (8) to the ordinary second-order differential equation

$$\frac{d^2\Phi_0}{dt^2} + \left(\frac{1}{t^2}(k_x^2 + k_y^2 + \xi) + k_z^2 + m^2\right)\Phi_0 = 0 \tag{10}$$

which can be solved in terms of Bessel functions (Lebedev, 1972)

$$\Phi_0 = \sqrt{t}Z_p(\sqrt{k_z^2 + m^2}t) \tag{11}$$

with

$$p = \tfrac{1}{2} + i\sqrt{k_x^2 + k_y^2 + \xi - 1/4} \tag{12}$$

Looking at the asymptotic behavior of the Hankel functions as $z \to \infty$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z-\nu\pi/2-\pi/4)}, \qquad H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z-\nu\pi/2-\pi/4)} \tag{13}$$

and the behavior of the Bessel function at $z = 0$

$$J_\nu(z) \sim \frac{z^\nu}{2^\nu\Gamma(\nu + 1)} \tag{14}$$

we have that the solution of equation (10), presenting an asymptotic behavior of the form (5), is

$$\Phi_{0(+\infty)}^{+} = \frac{C_\infty}{\sqrt{t}} H_p(\sqrt{k_z^2 + m^2}t) \tag{15}$$

where C_∞ is a normalization constant according to the standard inner product for the Klein–Gordon equation, the parameter p is given by (12), and the "+" indicates that the solution (15) is a positive-frequency mode for large values of the time.

Analogously, we have that in the vicinity of the singularity, looking at the quasiclassical solutions at $t = 0$, (6), the corresponding positive (+)- and negative (−)-frequency solutions take the form

$$\Phi_{0(0)}^{+} = \frac{C_0}{\sqrt{t}} J_p(\sqrt{k_z^2 + m^2}t), \qquad \Phi_{0(0)}^{-} = \frac{C_0}{\sqrt{t}} J_{-p}(\sqrt{k_z^2 + m^2}t) \tag{16}$$

Since we have been able to obtain single-particle states in the vicinity of $t = 0$ as well as in the asymptote $t \to \infty$, we can compute the density of particles created by the gravitational field with the help of the Bogoliubov coefficients (Grib *et al.*, 1988; Birrel and Davies, 1982). In the present case we do not need to compute the integral $|\beta_{kl}|^2 = |\langle\Phi_{k(+\infty)}^{+}, \Phi_{l(0)}^{-}\rangle|^2$ because of the recurrence relation existing between the Hankel and Bessel functions (Lebedev, 1972)

$$H_\nu^{(2)}(z) = i \cos ec(\nu\pi)(J_{-\nu}(z) - e^{\nu\pi i}J_\nu(z)) \tag{17}$$

Then we have that the positive-frequency solution $\Phi_{0(+\infty)}^{+}$ can be expressed in terms of $\Phi_{0(0)}^{+}$ and $\Phi_{0(0)}^{-}$ as follows:

$$\Phi_{0(+\infty)}^{+} = \mathfrak{C}(\Phi_{0(0)}^{+} - e^{p\pi i}\Phi_{0(0)}^{-}) = \alpha\Phi_{0(0)}^{+} + \beta\Phi_{0(0)}^{-} \tag{18}$$

where $\mathfrak{C}$ is a constant. From the normalization of the wave functions Φ we have that

$$|\alpha|^2 - |\beta|^2 = 1 \tag{19}$$

and, taking into account (17), we obtain

$$\left|\frac{\beta}{\alpha}\right|^2 = \exp[-2\pi \operatorname{Im}(p)] = \exp(-2\pi\sqrt{k_x^2 + k_y^2 + \xi - 1/4}) \tag{20}$$

Relation (20) shows (Mishima and Nakayama, 1988) that the density of scalar particles created is thermal.

Now, we proceed to discuss the process of creation of spin-1/2 particles in the cosmological background (1).

The Dirac equation in curved space can be written as

$$(\gamma^\alpha(\partial_\alpha - \Gamma_\alpha) + m)\Psi = 0 \tag{21}$$

where γ^α are the curved Dirac matrices, which satisfy the commutation relations $\{\gamma^\alpha, \gamma^\beta\}_+ = 2g^{\alpha\beta}$, and expressed in the diagonal tetrad gauge can be written in the form

$$\gamma^0 = \tilde{\gamma}^0, \qquad \gamma^1 = \frac{\tilde{\gamma}^1}{t}, \qquad \gamma^2 = \frac{\tilde{\gamma}^2}{t}, \qquad \gamma^3 = \tilde{\gamma}^3 \tag{22}$$

where $\tilde{\gamma}M^\alpha$ are the standard gamma matrices satisfying the relation $\{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\}_+ = 2\eta^{\alpha\beta}$. The Γ_α are the spin connections (Brill and Wheeler, 1957), which for diagonal metrics reduce to

$$\Gamma_\alpha = -\tfrac{1}{4} g_{\mu\beta} \Gamma^\beta_{\nu\alpha} S^{\mu\nu} \tag{23}$$

where $S^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Using the diagonal representation (22) for the curved Dirac matrices we obtain

$$\begin{aligned} \Gamma_0 &= 0, \qquad \Gamma_3 = 0 \\ \Gamma_1 &= \tfrac{1}{2}\tilde{\gamma}^0\tilde{\gamma}^1, \qquad \Gamma_2 = \tfrac{1}{2}\tilde{\gamma}^0\tilde{\gamma}^2 \end{aligned} \tag{24}$$

Since neither the Dirac matrices nor the metric depend on the space variables, it is possible to introduce the auxiliary spinor Ψ_0:

$$\Psi = t^{-1}\Psi_0(t)e^{i(k_x x + k_y y + k_z z)} \tag{25}$$

where the factor t^{-1} was introduced in (25) in order to cancel the contribution due to the spinor connections (24). Then, after substituting (25) into (24) we have

$$\left(\tilde{\gamma}^0\partial_t + \frac{i}{t}(\tilde{\gamma}^1 k_x + \tilde{\gamma}^2 k_y) + i\tilde{\gamma}^3 k_z + m\right)\Psi_0 = 0 \tag{26}$$

Notice that equation (26) is a system of coupled ordinary differential equations in the time variable. We can rewrite equation (26) as follows (Shishkin and Villalba, 1989)

$$(\hat{K}_1 + \hat{K}_2)\Phi = 0 \tag{27}$$

with

$$\hat{K}_2\Theta = (ik_x\tilde{\gamma}^1 + ik_y\tilde{\gamma}^2)\tilde{\gamma}^3\tilde{\gamma}^0\Theta = k\Theta \tag{28}$$

$$\hat{K}_1\Theta = t(\tilde{\gamma}^0\partial_t + i\tilde{\gamma}^3k_z + m)\tilde{\gamma}^3\tilde{\gamma}^0\Theta = -k\Theta$$

where $\Theta = \gamma^3\gamma^0\Psi_0$, and the constant of separation $k = i\sqrt{k_x^2 + k_y^2}$. The equation $\hat{K}_2\Theta = k\Theta$ establishes an algebraic relation among the components of the spinor Θ.

After introducing the following representation for the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{29}$$

we have that, with the help of equation (28), we reduce the problem of solving the Dirac equation (21) in the metric (1) to that of finding exact solutions of the coupled system of equations

$$\left(\frac{d}{dt} + \frac{k}{t}\right)\Theta_1 + (im - k_z)\sigma^3\Theta_2 \tag{30}$$

$$\left(\frac{d}{dt} - \frac{k}{t}\right)\Theta_2 + (im + k_z)\sigma^3\Theta_1 \tag{31}$$

where the spinor Θ has the block structure

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} \Theta_1 \\ -\dfrac{k_z + ik_y}{\sqrt{k_y^2 + k_x^2}}\sigma^3\Theta_1 \end{pmatrix} \tag{32}$$

with

$$\Theta_1 = \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} \tag{33}$$

Substituting (30) into (31) and vice versa, we get

$$\left(\frac{d^2}{dt^2} - \frac{k(k \pm 1)}{t^2} + (k_z^2 + m^2)\right)\Xi_{1,2} = 0 \tag{34}$$

with $k = i\sqrt{k_x^2 + k_y^2}$. This Bessel equation (34) has as solution the expression

$$\sqrt{t}Z_{k+1/2}(\sqrt{k_z^2 + m^2}t) \tag{35}$$

where the type of Bessel function to be considered depends on the value of t. For large values in time ($t \to \infty$), the positive-frequency solutions are

$$\begin{aligned} \Xi_{1(+\infty)}^{+} &= \sqrt{t}H_{k+1/2}^{(2)}(\sqrt{k_z^2 + m^2}t) \\ \Xi_{2(+\infty)}^{+} &= \frac{k_z + im}{\sqrt{k_z^2 + m^2}} \sqrt{t}H_{k-1/2}^{(2)}(\sqrt{k_z^2 + m^2}t) \end{aligned} \tag{36}$$

and in the vicinity of the initial singularity ($t \to 0$) we have

$$\begin{aligned} \Xi_{1(0)}^{+} &= \sqrt{t}J_{-k-1/2}(\sqrt{k_z^2 + m^2}t) \\ \Xi_{2(0)}^{+} &= -\frac{k_z + im}{\sqrt{k_z^2 + m^2}} \sqrt{t}J_{-k+1/2}(\sqrt{k_z^2 + m^2}t) \end{aligned} \tag{37}$$

In both cases, the choice of the modes was based on a comparison with the quasiclassical behavior given by equations (5) and (6).

Now, using the relation between the Hankel and Bessel functions (17), we can express Θ_1 with positive-frequency modes for large times ($t \to \infty$) in terms of Θ_1 in the vicinity of the time singularity

$$\Theta_{1(\infty)}^{+} = \mathfrak{D}(\Theta_{1(0)}^{+} - e^{k\pi i}\Theta_{0(0)}^{-}) = \alpha\Theta_{0(0)}^{+} + \beta\Theta_{0(0)}^{-} \tag{38}$$

where $\mathfrak{D}$ is a constant. Taking into account the normalization condition, we have that the coefficients α and β satisfy the relation

$$|\alpha|^2 + |\beta|^2 = 1 \tag{39}$$

and

$$\left|\frac{\beta}{\alpha}\right|^2 = \exp(2\pi ik) = \exp(-2\pi\sqrt{k_x^2 + k_y^2}) \tag{40}$$

showing that the distribution of spin-1/2 particles created by the background field (1) is thermal. Notice that the density of particles created can be obtained from (39) and (40), giving as result a Fermi–Dirac distribution.

$$n = |\beta|^2 = \frac{1}{\exp(2\pi\sqrt{k_x^2 + k_y^2}) + 1} \tag{41}$$

This result shows that the cosmological universe can create scalar as well as spinor particles in a thermal way. We have seen that the quasiclassical analysis developed in this article could be of help in the analysis of more realistic cosmological scenarios than the one used herein. It would be also possible to consider the corrections to the thermal spectrum when electromagnetic fields are present (Villalba, 1995). This will be considered in a forthcoming publication.

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